SMI-28-95

Fermi-Bose duality via extra dimension.

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February 1, 2008

Abstract

Representation of a D-dimensional fermion determinant as a path integral of exponent of a $(D+1)$-dimensional Hermitean bosonic action is constructed.

1 Introduction

In the recent paper [1] we proposed a method which allows to present a four dimensional fermion determinant as a path integral of exponent of a five dimensional constrained bosonic action. This method provides a path integral bosonization of fermion models in dimensions $D \geq 2$. The problem of bosonization of fermion models in dimensions $D > 2$ was studied previously by several authors (see for example [2], [3], [4], [5], [6]), however no quite satisfactory solution was known. Bosonic representation is particulary important for lattice gauge theories like lattice QCD as it allows to avoid complications related to integration over grassmanian variables. One way to handle the problem was proposed in papers [7], [8], where the algorithm for approximate inversion of the QCD fermion determinant was formulated replacing it by an ifinite series of bosonic determinants. In our paper [1] the exact effective bosonic action was explicitely constructed and it was proven that when the model is considered on the finite Euclidean lattice the path integral is convergent. Any gauge invariant observable in QCD may be calculated in this approach. However numerical simulations by Monte-Carlo method are not straightforward as the effective bosonic action is not Hermitean.

In the present paper I propose an alternative procedure which provides the representation of a D-dimensional fermion determinant in terms of a path integral of $(D+1)$-dimensional Hermitean bosonic action. This method seems to be more appropriate for Monte-Carlo simulations. The construction can be inverted providing

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the representation of bosonic determinant in terms of a path integral of fermionic effective action.

In the next section I illustrate the idea by the model on an infinite lattice. In the last section a corresponding construction for a finite lattice is presented.

2 Infinite lattice model

To illustrate the idea I firstly present a formal construction for the infinite lattice theory. In this formal proof I ignore the problem of convergence of the path integrals. In the next section a rigorous proof will be given for the case of a finite lattice.

Let $\det(B)$ be the determinant of some positive bounded Hermitean operator B.

$$\det(B) = \int \exp\{-a^D \sum_x [\bar{\psi}_m(x) B^{mn} \psi_n(x)]\} d\bar{\psi} d\psi \tag{1}$$

Here $\psi_m(x)$ is a fermion field, and m is a collective index which numerates spinorial, colour components e.t.c. These fields are defined on a D-dimensional Euclidean lattice with the lattice spacing a. Our goal is to construct a representation for the determinant of the operator B as a path integral of the exponent of a Hermitean bosonic action.

Let us introduce bosonic fields $\phi(x,t)$ which depend on one extra space coordinate t and have the same spinorial and internal structure as the fields ψ. The extra coordinate t is defined on the infinite one dimensional lattice with the lattice spacing b: $t = nb, \quad -\infty < n < +\infty$. The fields ϕ are described by a local Hermitean action

$$S = a^D b \sum_{n=-\infty}^{+\infty} \sum_x [i \frac{\phi^*_{n+1}(x) e^{iBb} \phi_n(x) - \phi^*_n(x) e^{-iBb} \phi_{n+1}(x)}{2b} \tag{2}$$

We shall prove the following identity

$$\det(B) = \lim_{b \to 0} \int \exp\{S + i(\phi^*_n(x)\chi(x) + \chi^*(x)\phi_n(x))]\} d\phi^*_n d\phi_n d\chi^* d\chi \tag{3}$$

Here $\chi(x)$ are D-dimensional bosonic fields which play the role of Lagrange multipliers imposing the constraints

$$\sum_n \phi_n(x) = \sum_n \phi^*_n(x) = 0. \tag{4}$$

R.h.s. of eq. (3) can be written in terms of eigenvectors of the operator B, B_α being the corresponding eigenvalues

$$I = \lim_{b \to 0} \int \exp\{b \sum_{n=-\infty}^{+\infty} \sum_\alpha [i \frac{\phi^{*\alpha}_{n+1} e^{iB^\alpha b} \phi^\alpha_n - \phi^{*\alpha}_n e^{-iB^\alpha b} \phi^\alpha_{n+1}}{2b} + \tag{5}$$

$$+ i(\phi^{*\alpha}_n \chi^\alpha + \chi^{*\alpha} \phi^\alpha_n)]\} d\phi^{*\alpha}_n d\phi^\alpha_n d\chi^{*\alpha} d\chi^\alpha.$$

To calculate the integral (5) we make the following change of variables:

$$\phi^\alpha_n \to \exp\{iB^\alpha nb\} \phi^\alpha_n, \quad \phi^{\alpha *}_n \to \exp\{-iB^\alpha nb\} \phi^{\alpha *}_n \tag{6}$$

After this transformation the integral (5) written in terms of Fourier components

$$\tilde{\phi}_k = b \sum_{n=-\infty}^{+\infty} \phi_n \exp\{-iknb\} \tag{7}$$

looks as follows

$$\begin{aligned} I = \lim_{b\to 0} \int \exp\{-(2\pi)^{-1} \sum_{\alpha} \int_{-\frac{\pi}{b}}^{+\frac{\pi}{b}} dk[|\tilde{\phi}_k|^2 \sin(kb)b^{-1} + \\ +i(\tilde{\phi}_k^{*\alpha}\chi^{\alpha} + \chi^{*\alpha}\tilde{\phi}_k^{\alpha})\delta(k - B_{\alpha})]\} d\tilde{\phi}_k^{*\alpha} d\tilde{\phi}_k^{\alpha} d\chi^{*\alpha} d\chi^{\alpha} \end{aligned} \tag{8}$$

Integrating over ϕ_α one gets

$$I = \lim_{b\to 0} Z \int \exp\{-\frac{1}{2\pi} \sum_{\alpha} \chi_\alpha^* [\sin(B_\alpha b)b^{-1}]^{-1} \chi_\alpha\} d\chi_\alpha^* d\chi_\alpha \tag{9}$$

where Z is the B_α independent (infinite) constant

$$Z = \int \exp\{-\frac{1}{2\pi} \sum_{\alpha} \int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} dk |\tilde{\phi}_\alpha(k)|^2 \sin(kb)b^{-1}\} d\tilde{\phi}_\alpha^*(k) d\tilde{\phi}_\alpha(k) \tag{10}$$

This constant may be eliminated by a proper normalization.

If the eigenvalues B_α are limited we can choose b sufficiently small, so that $B_\alpha b << 1$. Therefore we can replace $\sin(B_\alpha b)$ by $B_\alpha b$, and integrating over χ_α to get

$$I = \prod_{\alpha} B_\alpha = \det(B) \tag{11}$$

The eq.(3) is proven. The determinant of the positive bounded operator B is presented as the path integral of the purely bosonic Hermitean action.

However the derivation in this section was formal as we did not study the problem of convergence. In the next section I shall present an analogous construction for a finite lattice, where we are able to prove the convergence of all the integrals and make a representation like eq.(3) quite rigorous.

3 Finite lattice models. Lattice QCD

Having in mind applications to QCD in this section we consider two fermion flavours interacting vectorially with the Yang-Mills fields U_μ. The reason to consider two degenerate flavours is the positivity of the square of the gauge covariant Dirac operator. It allows to prove the convergence of the path integral for the model defined on the finite lattice. Formally the construction goes for any number of flavours, but a rigorous proof will be given for the case of even number of flavours. Firstly we reduce the problem to calculation of the determinant of Hermitean operator by using the identity

$$\det(\hat{D} + m) = \det[\gamma_5(\hat{D} + m)], \tag{12}$$

$$\hat{D} = \frac{1}{2}\gamma_\mu(D_\mu^* - D_\mu) \tag{13}$$

In eq.(13) D_μ is the lattice covariant derivative

$$D_\mu \psi(x) = \frac{1}{a}[U_\mu(x)\psi(x+a_\mu) - \psi(x)] \tag{14}$$

U_μ is a lattice gauge field.

It is convinient to present the square of fermion determinant in the following form

$$\int \exp\{a^4 \sum_{n=1}^{2} \sum_x \bar{\psi}_n(x)(\hat{D}+m)\psi_n(x)\} d\bar{\psi} d\psi = \det[\gamma_5(\hat{D}+m)]^2 = \tag{15}$$

$$= \int \exp\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\} d\bar{\psi} d\psi$$

The operator $-\hat{D}^2 + m^2$ is the square of Hermitean operator therefore all it's eigenvalues are positive .

We again introduce five dimensional bosonic fields $\phi(x,t)$with the same spinorial and internal structure as $\psi(x)$. The spatial components x are defined as above. The fifth component t to be defined on the one dimensional lattice of the length L with the lattice spacing b:

$$L = 2Nb, \quad -N < n \leq N \tag{16}$$

We choose $b << a$. The free boundary conditions in t are assumed:

$$\phi_n = 0, \quad n \leq -N, \quad n > N \tag{17}$$

The folowing identity is valid

$$\int \exp\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\} d\bar{\psi} d\psi = \tag{18}$$

$$= \lim_{L\to\infty, b\to 0} \int \exp\{a^4 b \sum_{n=-N+1}^{N} \sum_x [(2b^2)^{-1}(\phi^*_{n+1}(x) \exp\{i\gamma_5(\hat{D}+m)b\}\phi_n(x) +$$

$$+ \phi^*_n(x) \exp\{-i\gamma_5(\hat{D}+m)b\}\phi_{n+1}(x) - 2\phi^*_n\phi_n) +$$

$$+ \frac{i}{\sqrt{L}}(\phi^*_n(x)\chi(x) + \chi^*(x)\phi_n(x))]\} d\phi^*_n d\phi_n d\chi^* d\chi.$$

R.h.s. of eq. (18) can be written in terms of eigenvectors of the operator $\gamma_5(\hat{D}+m)$, which are denoted as D_α:

$$I = \lim_{L\to\infty, b\to 0} \int \exp\{b \sum_{n=-N+1}^{N} \sum_\alpha [\frac{\phi^{*\alpha}_{n+1} e^{iD^\alpha b}\phi^\alpha_n + \phi^{*\alpha}_n e^{-iD^\alpha b}\phi^\alpha_{n+1} - 2\phi^{*\alpha}\phi^\alpha}{2b^2} + \tag{19}$$

$$+ \frac{i}{\sqrt{L}}(\phi^{*\alpha}_n \chi^\alpha + \chi^{*\alpha}\phi^\alpha_n)]\} d\phi^{*\alpha}_n d\phi^\alpha_n d\chi^{*\alpha} d\chi^\alpha.$$

To check the convergence of the integral (19) we shall write it in terms of Fourier components. For this purpose it is convinient to replace the model on the lattice

with $2N$ points and free boundary conditions by the equivalent model on the lattice with $2N+1$ points, periodic boundary conditions and the additional constraint

$$\phi_{-N} = \phi^*_{-N} = 0 \tag{20}$$

These constraints may be imposed by adding to the action the corresponding terms with Lagrange multipliers λ. Then the integral (19) may be written in terms of Fourier components

$$\tilde{\phi}_k = \sum_{-N}^{+N} \phi_n \exp\{\frac{2i\pi kn}{2N+1}\} \tag{21}$$

as follows

$$I = \lim_{L\to\infty, b\to 0} \int \exp\{b(2N+1)^{-1} \sum_\alpha [\sum_{k=-N}^{N} |\tilde{\phi}^\alpha_k|^2 [\cos(\frac{2\pi k}{2N+1} - D^\alpha b) - 1](2b^2)^{-1} + \tag{22}$$

$$+ib(L)^{-\frac{1}{2}}(\tilde{\phi}^{*\alpha}_0 \chi^\alpha + \chi^{*\alpha}\tilde{\phi}^\alpha_0) + ib(2N+1)^{-1} \sum_k (\tilde{\phi}^{*\alpha}_k \exp\{\frac{2i\pi kN}{2N+1}\}\lambda_\alpha +$$

$$+\lambda^*_\alpha \tilde{\phi}^\alpha_k \exp\{\frac{-2i\pi kN}{2N+1}\})]\} d\tilde{\phi}^*_\alpha(k) d\tilde{\phi}_\alpha(k) d\chi^*_\alpha d\chi_\alpha d\lambda^*_\alpha d\lambda_\alpha$$

The representation (22) makes the convergence of the integral obvious.

Now we come back to coordinate space and to calculate the integral (19) we make the following change of variables:

$$\phi^\alpha_n \to \exp\{iD^\alpha nb\}\phi^\alpha_n, \quad \phi^{\alpha*}_n \to \exp\{-iD^\alpha nb\}\phi^{\alpha*}_n \tag{23}$$

$$\lambda^\alpha \to \exp\{iD^\alpha Nb\}\lambda^\alpha, \quad \lambda^{*\alpha} \to \exp\{-iD^\alpha Nb\}\lambda^{*\alpha}$$

Then one gets

$$I = \lim_{L\to\infty, b\to 0} \int \exp\{b \sum_{n=-N}^{+N} [\frac{\phi^{*\alpha}_{n+1}\phi^\alpha_n + \phi^{*\alpha}_n \phi^\alpha_{n+1} - 2\phi^{*\alpha}_n \phi^\alpha_n}{2b^2} + \tag{24}$$

$$+i(L)^{-\frac{1}{2}}(\phi^{*\alpha}_n e^{iD^\alpha nb}\chi^\alpha + \chi^{*\alpha} e^{-iD^\alpha nb}\phi^\alpha_n)] +$$

$$+i(\phi^{*\alpha}_{-N}\lambda^\alpha + \lambda^{*\alpha}\phi^\alpha_{-N})\} d\phi^{*\alpha}_n d\phi^\alpha_n d\chi^{*\alpha} d\chi^\alpha d\lambda^{*\alpha} d\lambda^\alpha$$

Now the quadratic form in the exponent does not depend on D_α and therefore the corresponding determinant is a trivial constant. So we can calculate the integral by finding the stationary point of the exponent, which is defined by the following equations:

$$b^{-2}(\phi^{*\alpha}_{n+1} + \phi^{*\alpha}_{n-1} - 2\phi^{*\alpha}_n) + iL^{-\frac{1}{2}}\chi^{*\alpha} e^{-iD_\alpha nb} = 0, \quad n \neq -N$$

$$b^{-2}(\phi^\alpha_{n+1} + \phi^\alpha_{n-1} - 2\phi^\alpha_n) + iL^{-\frac{1}{2}}\chi^\alpha e^{iD_\alpha nb} = 0, \quad n \neq -N \tag{25}$$

$$b^{-2}(\phi^{*\alpha}_{-N+1} - 2\phi^{*\alpha}_{-N}) + iL^{-\frac{1}{2}}\chi^{*\alpha} e^{iD_\alpha Nb} + i\lambda^{*\alpha} = 0$$

$$b^{-2}(\phi^\alpha_{-N+1} - 2\phi^\alpha_{-N}) + iL^{-\frac{1}{2}}\chi^\alpha e^{-iD_\alpha Nb} + i\lambda^\alpha = 0$$

$$\phi_{-N} = \phi^*_{-N} = 0, \quad \phi_{N+1} = \phi^*_{N+1} = 0$$

For small b these equations can be approximated by the differential equations:

$$\ddot{\phi}^{*\alpha} + iL^{-\frac{1}{2}}\chi^{*\alpha}e^{-iD^\alpha t} = 0 \tag{26}$$

$$\ddot{\phi}^{\alpha} + iL^{-\frac{1}{2}}\chi^{\alpha}e^{iD^\alpha t} = 0$$

$$\phi(\frac{L}{2}) = \phi(-\frac{L}{2}) = 0, \quad \phi^*(\frac{L}{2}) = \phi^*(-\frac{L}{2}) = 0$$

The solution of these eq.s is

$$\phi^{*\alpha} = \frac{i\chi^{*\alpha}}{\sqrt{L}(D^\alpha)^2}e^{-iD^\alpha t} - \tag{27}$$

$$-\frac{\chi^{*\alpha}}{(D^\alpha)^2\sqrt{L}}[\frac{2t}{L}\sin(\frac{D^\alpha L}{2}) + i\cos(\frac{D^\alpha L}{2})]$$

$$\phi^{\alpha} = \frac{i\chi^{\alpha}}{\sqrt{L}(D^\alpha)^2}e^{iD^\alpha t} +$$

$$+\frac{\chi^{\alpha}}{(D^\alpha)^2\sqrt{L}}[\frac{2t}{L}\sin(\frac{D^\alpha L}{2}) - i\cos(\frac{D^\alpha L}{2})]$$

Substituting this solution to the eq.(24) we get

$$I = \lim_{L\to\infty}\int \exp\{-\frac{-\chi^{*\alpha}\chi^\alpha}{(D^\alpha)^2} + O(L^{-1})\} = \prod_\alpha (D^\alpha)^2 = \det(-D^2 + m^2) \tag{28}$$

The eq.(18) is proven. It is clear from eqs.(27,28), that in the limit $L \to \infty$ the dependence on the boundary conditions disappear.

For practical calculations it may be useful to linearize the eq.(18). It can be done in a straightforward way using the fact that the operator $\gamma_5(\hat{D} + m)$ is bounded [7]:

$$||\gamma_5(\hat{D} + m)|| \leq 8a^{-1} + m \tag{29}$$

. Therefore for any α the product $D^\alpha b << 1$. The exponent $\exp\{ibD_\alpha\}$ may be replaced by $1 + ibD_\alpha - \frac{1}{2}b^2(D_\alpha)^2$ and linearized version of the eq.(18) looks as follows

$$\int \exp\{a^4 \sum_x \bar{\psi}(x)(\hat{D}^2 - m^2)\psi(x)\}d\bar{\psi}d\psi = \tag{30}$$

$$= \lim_{L\to\infty, b\to 0}\int \exp\{a^4 b \sum_{n=-N+1}^{N}\sum_x [(2b^2)^{-1}(\phi^*_{n+1}(x)\phi_n(x) + \phi^*_n(x)\phi_{n+1}(x) - 2\phi^*_n\phi_n) +$$

$$+i[\phi^*_{n+1}(x)\gamma_5(\hat{D} + m)\phi_n(x) - \phi^*_n(x)\gamma_5(\hat{D} + m)\phi_{n+1}(x)][2b]^{-1} +$$

$$+\frac{1}{2}\phi^*_n(D^2 - m^2)\phi_n + \frac{i}{\sqrt{L}}(\phi^*_n(x)\chi(x) + \chi^*(x)\phi_n(x))]\}d\phi^*_n d\phi_n d\chi^* d\chi.$$

One can get rid off the last term in the exponent by solving explicitely the constraints

$$\sum_{n=-N+1}^{N}\phi_n(x) = 0 \tag{31}$$

Substituting the solution of this equation to the integral (30) we get a representation for the determinant of the square of the gauge covariant Dirac operator as the path integral of the exponent of the purely bosonic Hermitean action.

4 Discussion

Comparing the results presented above with the previous ones [1] we note that the new bosonic representation for gauge covariant Dirac operator is based on the Hermitean action and therefore is better suited for numerical simulations. We also got rid off the extra parameter λ which was present in our first paper. One can hope that the construction given in this paper will allow to use directly the Monte-Carlo method for calculation of fermion determinant.

Acknowledgements.
This researsh was supported in part by Russian Basic Research Fund under grant 94-01-00300a.

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